

On the Scattering Theory and the Integrable Ultradiscrete System

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(Received November 30, 2001)

The outline of the S-matrix theory of the scattering theory and the integrable ultradiscrete system (the soliton cellular automaton and the box and ball system) are given. Both resemblances, correspondence and some simulation are shown.

The structures of particles and materials have been studied due to the scattering in the many cases of the macro-system and the micro-system. For example the observation of scattering of α -particles by a thin gold-foil in 1911 is the famous experiment of Rutherford, which was the splendid step for understanding of the structure of the atom. This scattering experiment showed the model of atom that a small nucleon with positive charge exists at the center of the atom. There are various scattering theories and approximations in classical and quantum dynamics, one of common formulas is the S-matrix theory in the Hamiltonian system. Scattering problem is considered as a transition to the final state after the scattering from initial state before the scattering. In general initial and final states on scattering process are supposed to be asymptotic states which exist at infinite distance in the past and the future of the infinity and do not interact with each other.

Mathematically scattering problems are treated as the spectrum theory of the Schrödinger operator and/or the problems of partial differential equations. The scattering state is defined as the state which is operated on the initial state by the scattering operator. This scattering operator is presented as the product of wave operators.

where W^\dagger is Hermitian conjugate of W . S is the operator which transfers the initial state $\phi(t)$ of the infinite past $t = -\infty$ to the final state $\psi(t)$ of the infinite future $t = \infty$:

$$S\phi(-\infty) = \psi(\infty), \quad (11)$$

and commute to H_0 :

$$[S, H_0] = SH_0 - H_0S = 0. \quad (12)$$

One of the mathematical problems is the proof of the completeness of wave operators. It is complete, if following equation is satisfied

$$W_+H_0 = W_-H_0 = H, \quad (13)$$

which is equivalent to the unitarity of S-matrix²⁾ :

$$|S| = SS^\dagger = 1. \quad (14)$$

Assuming that the Hamiltonian can be written as

$$H = H_0 + V, \quad (15)$$

where V would be zero in the absence of scatterer³⁾. The transition operator T is defined as using this potential V

$$V\psi = T\phi, \quad (16)$$

and satisfied with the following equation

$$T = V + V \frac{1}{E + i\epsilon - H_0} T, \quad (17)$$

where ϵ is a arbitrary small value. The formal solution of this equation is

$$T = V + V \frac{1}{E + i\epsilon - H} V. \quad (18)$$

In terms of the state

$$\psi = \phi + \frac{1}{E + i\epsilon - H} V\psi, \quad (19)$$

this equation is called Lippmann-Schwinger equation.

Phenomenologically, the scattering can be classified roughly into two types: elastic scattering and inelastic ones. In the former the energy and/or components of the

H is the Hamiltonian of the system, H_0 is the Hamiltonian of the unperturbed free system and t is a time variable, then the time evolution of the system is described as e^{itH} . The initial value problem of the Schrödinger equations are

$$i \frac{d}{dt} \psi(t) = H\psi(t), \quad \psi(0) = \psi_0, \quad (1)$$

$$i \frac{d}{dt} \phi(t) = H_0\phi(t), \quad \phi(0) = \phi_0, \quad (2)$$

and their solutions are

$$\psi(t) = U(t)\psi_0, \quad (3)$$

$$\phi(t) = U_0(t)\phi_0, \quad (4)$$

where

$$U(t) = e^{-itH}, \quad (5)$$

$$U_0(t) = e^{-itH_0}, \quad (6)$$

operator $U(t)$ is a time develop operator, which translate a state to the state at time t after. This $U(t)$ is unitary operator satisfied following relation

$$U(t)U(s) = U(t+s), \quad U(-t) = U(t)^{-1}. \quad (7)$$

When ψ_0 is a scattering state of H , initial state of $\psi(t) = U(t)\psi_0$ is ψ_- and $\psi(t)$ behave like free motion with initial state ψ_- at $t \mapsto -\infty^1$:

$$U(t)\psi_0 \sim U_0(t)\psi_- \quad t \mapsto -\infty. \quad (8)$$

The wave operator W_- makes the initial state ψ_- to correspond with ψ_0 , similarly W_+ makes the final state ψ_+ to correspond with ψ_0 . They are defined as:

$$W_{\pm} = W_{\pm}(H, H_0) = s\text{-}\lim_{t \rightarrow \pm\infty} U(-t)U_0(t), \quad (9)$$

where $s\text{-lim}$ is a strong convergence which take a limit after operating to a vector of Hilbert space. The scattering operator S is defined as follows:

$$S = W_+^{\dagger}W_-, \quad (10)$$

system do not change during the scattering, but in the latter the energy of the final outgoing is different from the one of the initial incoming, and/or other things between before and after of scattering change. In compound particle and material phenomenally there are rearrangement scattering, resonance scattering, capture scattering, etc.

Many bodies problems of scattering phenomena come to the two body scattering in the many cases. The scattering of two nonrelativistic particles with no structure in the potential depended only on the distance between two particles is equivalent to the one body problem, and this is a example of elastic scattering. On the other hand the scatterings of the particles with structure include elastic and inelastic scattering, and the resonance and/or rearrangement scattering are often observed in atomic, nuclear, and particle physics³⁾.

For example let us consider following scattering of two particles



where C is a composite resonance particle, D and F are particles which have different components and structure from A and B ^{4,5)}.

H is the Hamiltonian of the system and is written as two type of before and after of scattering

$$H = H_{0i} + V_i = H_{0f} + V_f, \quad (21)$$

$$H_{0i} = H_A + H_B, \quad (22)$$

$$H_{0f} = H_D + H_F, \quad (23)$$

$$H_{0i}\phi_- = E_i\phi_-, \quad (24)$$

$$H_{0f}\phi_+ = E_f\phi_+, \quad (25)$$

where V_i is the interaction potential between A and B , E_i is the energy of initial state, V_f is the interaction potential between D and F and E_f is the energy of final state. Therefore

$$\psi_i = \phi_i + \frac{1}{E_i + i\epsilon - H} V_i \psi_i, \quad (26)$$

$$\psi_f = \phi_f + \frac{1}{E_f + i\epsilon - H} V_f \psi_f. \quad (27)$$

Suppose a finite-range potential V is attractive, in the case of a finite barrier, the particle can be trapped inside, but not forever. Such a trapped state has a finite life time, which is called a quasis-bound state³⁾. Thus resonance scatterings occur through metastable state or virtual state⁵⁾. These metastable states may be regarded as the resonance states.

In the previous paper⁶⁾ it is shown that the resonance particle in the quark model is analogous to the intermediate combined state of the scattering in the box and ball system, which is one of the integrable ultradiscrete system. In this article we want to take roughly correspondence between above scattering theory and the integrable ultradiscrete system.

The soliton cellular automaton(SCA) proposed by Takahashi and Satsuma⁷⁾ is one of the integrable ultradiscrete systems and is equivalent to the box and ball system (BBS). The element of the SCA is expressed as the following⁸⁾

$$U_j^{t+1} = \min(1 - U_j^t, \sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1}). \quad (28)$$

where U_j^t of 1+1 dimension has two valued 1 or 0. If index t denotes a time variable, the state at t , which is sum of space index j , may be corresponded to $\psi(t)$ at that time:

$$\sum_{j=-\infty}^{\infty} U_j^t \sim \psi(t). \quad (29)$$

Practically this is approximated by

$$\sum_{j=-M}^M U_j^t \quad \text{or} \quad \sum_{j=0}^N U_j^t, \quad (30)$$

where M and N are appropriate numbers. So the time evolution of this system is described as

$$\sum_{j=0}^N U_j^{t+1} - \sum_{j=0}^N U_j^t, \quad (31)$$

which is one step of the time evolution and corresponds to a discretized process of $i \frac{d}{dt} \psi(t) = H\psi(t)$. And this step corresponds the one routine of the evolution rule of the BBS, as following

- (1) Move every 1 (ball) only once at one step ($t \rightarrow t + 1$).
- (2) Exchange the leftmost 1 (move ball) with its nearest right 0 (into empty box).

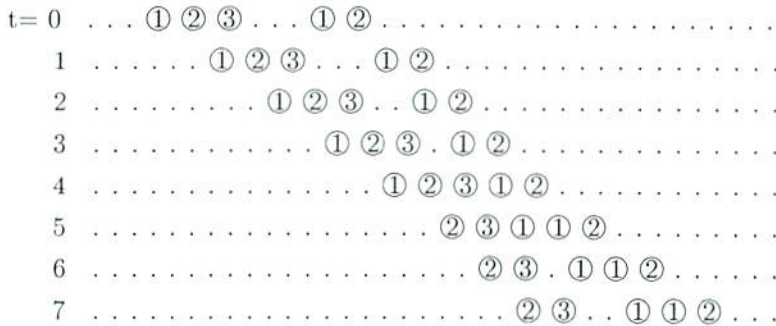
In the extended box and ball system (EBBS), the capacity of the box and the number of kinds of balls are permitted to be plural, so the time evolution rule is described as the following⁹⁾

$$U_{n,j}^t = \min\left(\sum_{m=-\infty}^{n-1} U_{m,j}^{t-1} - \sum_{m=-\infty}^{n-1} U_{m,j}^t, L - \sum_{i=1}^{j-1} U_{n,i}^t - \sum_{i=j}^M U_{n,i}^{t-1}\right), \quad (32)$$

where L is the capacity of the box and M is the number of kinds of balls. In this case the practical rule from t to $t + 1$ is

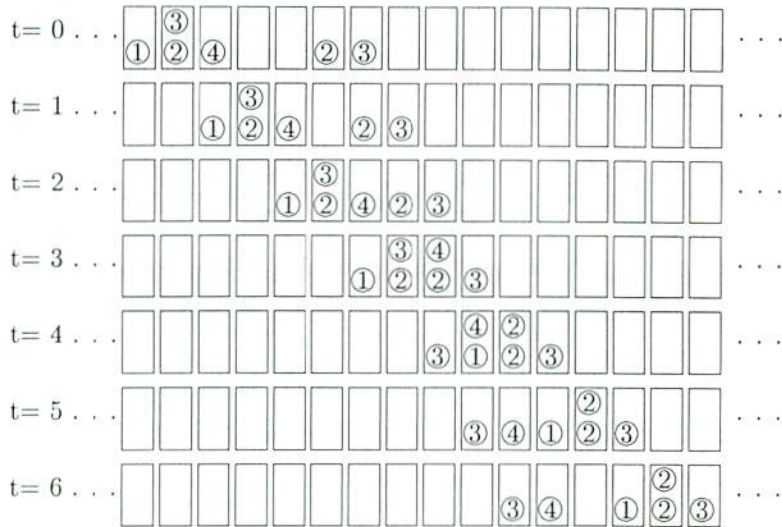
- (1) Move every ball only once.
- (2) Move the leftmost ball-1 to the nearest right box with space, i.e., to the nearest right box which contains less than L balls.
- (3) Move the leftmost ball-1 among the rest to its nearest right box with space.
- (4) Repeat this procedure until all ball-1 has been moved.
- (5) Continue the same procedure (2)-(4) for ball-2, ball-3, ..., ball- M .
- (6) Repeat this procedure successively until all of the balls have been moved.

An example of the case which L is one and M is three (ball-1, ball-2 and ball-3 are presented as ①, ② and ③ respectively), is following



Assuming that a series of balls corresponds to a particle and/or matter, which is considered as A, B , etc. of the eq.(20), and balls are regarded as its components, the state at $t = 4, 5$ can be presumed the resonant state of the rearrangement scattering.

The next case is the example of $L = 2$ and $M = 4$.



In these cases interaction force in a short-range potential V may be produced by exchanging components. If the scattering is defined as time development with interaction, various phenomena are regarded as the scattering problems. These integrable ultradiscrete systems are so useful tool as the simulation algorithm.

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